

PROPAGATION OF A LAMINAR JET OF ELECTRICALLY CONDUCTING FLUID
IN A UNIFORM MAGNETIC FIELD

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Several papers [1-4] have considered the propagation of a plane laminar jet of incompressible conducting fluid in a uniform magnetic field for magnetic Reynolds numbers much less than unity. These papers have investigated the flow of a free jet in a transverse magnetic field for small values of the magnetic interaction parameter. Equations for the first approximations were obtained in [1, 2] by a series expansion in the small interaction parameter close to the ordinary solution (without magnetic field) for the jet. The equations for the zero-th and first approximations were integrated in [3]. The same author also found a similar solution for a turbulent jet, the turbulent transfer coefficient being chosen according to Prandtl's method. As regards the solution found in [4], it suffers from the defect that the constant of integration which connects the real velocity profiles with those found in the paper remains undetermined. The present paper gives an approximate solution of the same dynamic problem of the propagation of a free plane jet in a uniform field, no assumption being made as to the smallness of the interaction parameter. In order to do this the integral method of solution, common in ordinary hydrodynamics [5, 6] is employed. The solution of the problem is generalized to include the case of a finite value of the Hall parameter.

1. Let a jet of conducting fluid flow from an infinite thin slit located at the point 0 in the direction of the x axis, into a space filled with the same fluid at rest (Fig. 1). We shall assume that the region of jet propagation is situated in an infinite external uniform magnetic field of field strength H_0 , in the direction of the y axis and with a magnetic Reynolds number $R_m \ll 1$.

In the case under consideration the equations of motion and continuity, and also the boundary conditions have the form

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} - \sigma \frac{\mu^2 H_0^2}{\rho} u, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1.1)$$

$$\frac{\partial u}{\partial y} = 0 \quad \text{at } y=0, \quad u = \frac{\partial u}{\partial y} = 0 \quad \text{at } y = \pm \infty. \quad (1.2)$$

In accordance with the integral method of solution the velocity profile will be sought in the form of a fourth-degree polynomial

$$u = a_0 + a_2 y^2 + a_4 y^4, \quad (1.3)$$

where the odd terms have been omitted in view of the symmetry of the situation.

In order to determine the coefficients of the polynomial (1.3) we require that the following boundary conditions be fulfilled:

$$\nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma \mu^2 H_0^2}{\rho} u = u_m \frac{du_m}{dx} \quad \text{at } y=0, \\ u = \frac{\partial u}{\partial y} = 0 \quad \text{at } y = \pm \infty. \quad (1.4)$$

The condition for $y = 0$ is obtained from the initial equation (1.1), u_m is the value of the velocity on the axis of the jet [$u_m = u(x, 0)$], δ is the effective width of the jet.

Determining the constants a_0 , a_2 , a_4 and setting them in the relation (2.3), we find an expression for the required velocity profile

$$u = -\frac{\delta^2}{4} \left(\frac{du_m}{dX} + N \right) u_m F(\varphi), \quad F(\varphi) = 1 - 2\varphi^2 + \varphi^4, \\ \varphi = \frac{y}{\delta}, \quad X = xv, \quad N = \frac{\sigma \mu^2 H_0^2}{\rho \nu}. \quad (1.5)$$

In order to determine the value of the velocity on the axis of the jet u_m and its thickness δ we must have two equations. The first of these comes from the expression for the velocity (1.5) on the axis of the jet

$$1 = -\frac{\delta^2}{4} \left(\frac{du_m}{dX} + N \right). \quad (1.6)$$

The second equation is obtained from the initial equations (1.1) by integrating them across the jet

$$\frac{d}{dx} (u_m^2 \delta) + \frac{21}{16} N u_m \delta = 0. \quad (1.7)$$

The system of equations (1.6) and (1.7) gives us an expression for the maximum velocity

$$u_m = \delta \left(\frac{11}{16} N + \frac{8}{\delta^2} \right) \frac{dX}{d\delta}. \quad (1.8)$$

The effective thickness of the jet δ is determined from the equation

$$\delta \left(8 + \frac{11}{16} N \delta^2 \right) \frac{d^2 \delta}{dX^2} + \left(4 - \frac{27}{16} N \delta^2 \right) \left(\frac{d\delta}{dX} \right)^2 = 0, \quad (1.9)$$

with the boundary condition $\delta = 0$ for $x = 0$. We find an expression for the thickness of the jet by integrating equation (1.9):

$$\int_0^\delta \frac{\sqrt{\delta} d\delta}{(1 + \frac{11}{128} N \delta^2)^{3/4}} = C \nu x. \quad (1.10)$$

In order to determine the constant of integration C we integrate equation (1.7) obtaining

$$u_m^2 \delta + \frac{21}{16} N \int_0^\delta u_m \delta dx = \text{const} = I_0/k \\ \left(k = \int_{-1}^1 F^2 d\varphi = \frac{128}{315} \right). \quad (1.11)$$

The constant of integration is taken to be equal to I_0 since the term $u_m^2 \delta k$ determines the initial momentum of the jet I_0 for $x = 0$.

Using relations (1.10) and (1.11) we determine the constant C:

$$C = 8\sqrt{k} / \sqrt{I_0}. \quad (1.12)$$

The relations obtained enable us to determine the longitudinal (u) and transverse (v) velocity components by using Eq. (1.8) and the continuity equation.

It is clear from the expression for jet thickness δ that for values of the magnetic interaction parameter which are nonzero ($N \neq 0$) there ceases to be any development of the jet at a certain finite distance from the source equal to $x \approx 1.35 N^{-3/4} \sqrt{I_1/v^2}$. Here the jet thickness becomes infinitely large (Fig. 2), while the velocity becomes zero. The jet momentum decreases along the x axis at a rate which increases as the parameter N increases; the rate of flow of the jet which is proportional to the quantity $u_m \delta$, passes through an extremum as the distance from the jet source ($u_m = 0$ for $x = 0$) increases, and becomes zero once more for $x \approx 1.35 N^{-3/4} \sqrt{I_1/v^2}$. The transverse velocity component changes sign at a value of x corresponding to the maximum value of the flowrate. Consequently, the decrease in flowrate of the stream is associated with the fluid in the jet being forced out into the surrounding medium.

We note that in the region where there is a sharp increase in jet thickness we must allow for effects arising from the fact that the transverse and longitudinal velocity components are commensurable, and also for the formation of a pressure gradient associated with the distortion of the streamlines. In other words, we must exercise special care when applying the boundary layer equations to this region.

2. In order to compare the solution obtained by the integral method with the exact solution (in the framework of the asymptotic layer theory) we shall consider the propagation of a jet in a nonuniform field.

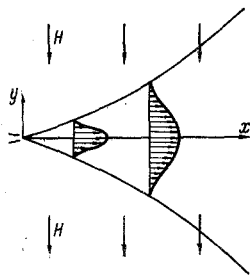


Fig. 1

As in [7], we shall assume that the magnetic field varies inversely as the width of the jet:

$$H = H_0 / \delta. \tag{2.1}$$

We have from the system of equations (1.6) and (1.7), taking (2.1) into account, the following expressions for the required quantities:

$$u = u_m F(\varphi), \quad F(\varphi) = 1 - 2\varphi^2 + \varphi^4, \tag{2.2}$$

$$u_m = \frac{1}{C^2} \left(12 + \frac{3}{8} N \right) x^\alpha,$$

$$\delta = Cx^\beta, \quad \alpha = -\frac{4+N}{12+3/8N}, \tag{2.3}$$

$$\beta = \frac{8 \pm 11/16N}{12+3/8N} \quad \left(N = \frac{\sigma \mu^2 H_0^2}{\rho \nu C^2} \right).$$

In order to determine the constant C we make use of the fact that an integral of the form

$$\int_{-\delta}^{\delta} u^{-\beta/\alpha} dy = D \tag{2.4}$$

does not change along the axis of the stream. From this condition we obtain for C

$$C = D^\alpha \left[\left(12 + \frac{3}{8} N \right) \left(\int_{-1}^1 |F(\varphi)|^{-\beta/\alpha} d\varphi \right) \right]^{-\alpha}. \tag{2.5}$$

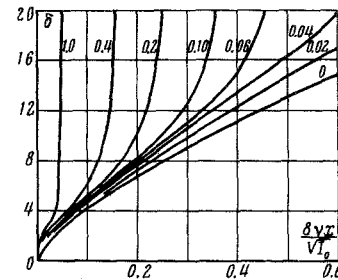


Fig. 2

Similar results obtained from the exact solution of the problem [7] are quoted at this point:

$$\alpha = -\frac{1}{3} \left(1 + \frac{N}{2} \right), \quad \beta = -\frac{2}{3} \left(1 + \frac{N}{8} \right) \tag{2.6}$$

$$\left(N = \frac{\sigma \mu^2 H_0^2}{\rho \nu b^2} \right),$$

$$b = \frac{D^{-\alpha}}{\sqrt{6\nu}} \left(\sqrt{6\nu} \int_{-\infty}^{\infty} (\operatorname{ch} \varphi)^{\frac{2\beta}{\alpha}} d\varphi \right)^{-\alpha} \quad (\varphi = bx^{\beta/2}y).$$

In comparing the solution obtained by the integral method with the exact solution we must allow for the difference in the formulas defining the parameter N .

Figure 4 compares the self-similarity constants α and β as functions of the magnetic interaction parameter, where N is calculated from formula (2.6). It is clear from the figure that the results of the integral method (continuous line) relating to the variation of the maximum velocity and jet thickness along its axis are in agreement with similar results of the exact solution (broken line). From the solutions which have been obtained it follows that a self-similar solution for the jet exists when the magnetic parameter varies within the region $0 \leq N < 3.4$. The upper limit of the parameter $N = 3.4$ is determined by the fact that the flowrate of the jet along the axis ought to increase in the case under consideration. Here the self-similarity constants vary, respectively, within the limits

$$-1/3 \geq \alpha > 1, \quad -2/3 \geq \beta > -1.$$

It also follows from the solutions that for self-similar spreading of the jet in a magnetic field a specific relationship must hold between the dynamic jet characteristic D and the applied magnetic field.

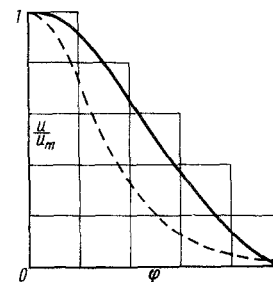


Fig. 3

Physically, it is clear that this is connected with the fact that for given values of the initial jet momentum and the parameter which

determines how the magnetic field varies along the axis of the jet, self-similar development of the jet occurs only for one characteristic value of the magnetic field strength H_0 .

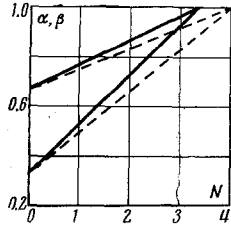


Fig. 4

For the limiting value of the parameter $N = 3.4$, $\alpha = \beta = -1$ the jet degenerates to flow at a constant flowrate; consequently, the fluid flowrate for the jet source is equal to zero [4]. We note here that the solution of [8] refers to just this case, which to our minds seems physically unrealistic.

Dimensionless velocity profiles are given for transverse cross-sections in Fig. 3. The comparison shows clearly that the velocity distributions obtained by the integral method (continuous line) differ markedly from the exact solution (broken line). We note that in the case under consideration the self-similar velocity profile obtained by the integral method has the same form as for purely hydrodynamic jet flow.

3. We shall consider the problem of jet propagation in a uniform magnetic field allowing for the dependence of the conductivity on the magnetic field, and for the same condition $R_m \ll 1$. The initial system of equations is written in the following form [9, 10] for a finite value of the Hall parameter $\omega\tau$:

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= v \frac{\partial^2 u}{\partial y^2} - \frac{\sigma \mu^2 H_0^2}{\rho(1 + \omega^2 \tau^2)} (u - \omega \tau w), \\ u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} &= v \frac{\partial^2 w}{\partial y^2} - \frac{\sigma \mu^2 H_0^2}{\rho(1 + \omega^2 \tau^2)} (w + \omega \tau u), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned} \quad (3.1)$$

In what follows we neglect the term $\omega\tau w$ in the first equation, as was done, for example, in [9, 10]. We shall seek the velocity profiles in the form of polynomials

$$u = a_0 + a_2 y^2 + a_4 y^4, \quad w = b_0 + b_2 y^2 + b_4 y^4. \quad (3.2)$$

Determining the coefficients from the six boundary conditions

$$\begin{aligned} u_m \frac{du_m}{dx} &= v \frac{\partial^2 u}{\partial y^2} - N^\circ u_m \quad \text{at } y=0, \\ u_m \frac{dw_m}{dx} &= v \frac{\partial^2 w}{\partial y^2} - N^\circ (w_m - \omega \tau u_m), \\ u = w = 0, \quad \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} &= 0 \quad \text{at } y=\delta, \end{aligned} \quad (3.3)$$

we obtain

$$\begin{aligned} u &= -u_m \left(\frac{du_m}{dx} + N^\circ \right) \frac{\delta^2}{4} F(\varphi), \quad F(\varphi) = 1 - 2\varphi^2 + \varphi^4, \\ w &= -\frac{\delta^2}{4} \left[u_m \frac{du_m}{dx} + N^\circ (w_m + \omega \tau u_m) \right] F(\varphi). \end{aligned} \quad (3.4)$$

Here N° denotes the interaction parameter

$$N^\circ = \frac{\sigma \mu^2 H_0^2}{\rho v (1 + \omega^2 \tau^2)}. \quad (3.5)$$

We have the following equations for determining u_m and w_m :

$$\begin{aligned} 1 &= -\frac{\delta^2}{4} \left(\frac{du_m}{dx} + N^\circ \right), \\ w_m &= -\frac{\delta^2}{4} \left[u_m \frac{du_m}{dx} + N^\circ (w_m + \omega \tau u_m) \right]. \end{aligned} \quad (3.6)$$

The solution of the first equation of (3.6) and Eq. (1.7) is given above (with N changed to N°). The system of equations (3.6) gives us an expression for the maximum value of the transverse velocity component

$$w_m = Cu - N \omega \tau u_m \int_0^x \frac{dx}{u}. \quad (3.7)$$

The constant of integration $C = 0$, since by hypothesis the jet has no initial rotation relative to the y axis ($uw\delta = 0$ for $x = 0$).

Calculating the integral in expression (3.7), we have

$$w_m = -\frac{8}{11} \omega \tau u_m \ln(1 + \frac{11}{64} N \delta^2). \quad (3.8)$$

The solution which has been obtained shows that a motion transverse to the plane of flow arises in the jet. As $\omega\tau$ increases the velocity of the sideways flow increases. However, a further increase of $\omega\tau$, starting from some particular value, leads to a decrease of velocity of transverse motion until $w \rightarrow 0$ for $\omega\tau \rightarrow \infty$. The longitudinal velocity component u decreases as $\omega\tau$ increases, and as $\omega\tau \rightarrow \infty$ it tends to the limiting velocity distribution for a flow of nonconducting fluid (case $N = 0$).

The solutions obtained for the propagation of a conducting fluid jet in the absence of a magnetic field ($N = 0$) pass into the corresponding solutions for purely hydrodynamic flow.

It should be noted that the integral method has been applied to the problem under consideration by E. V. Shcherbina [11].

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